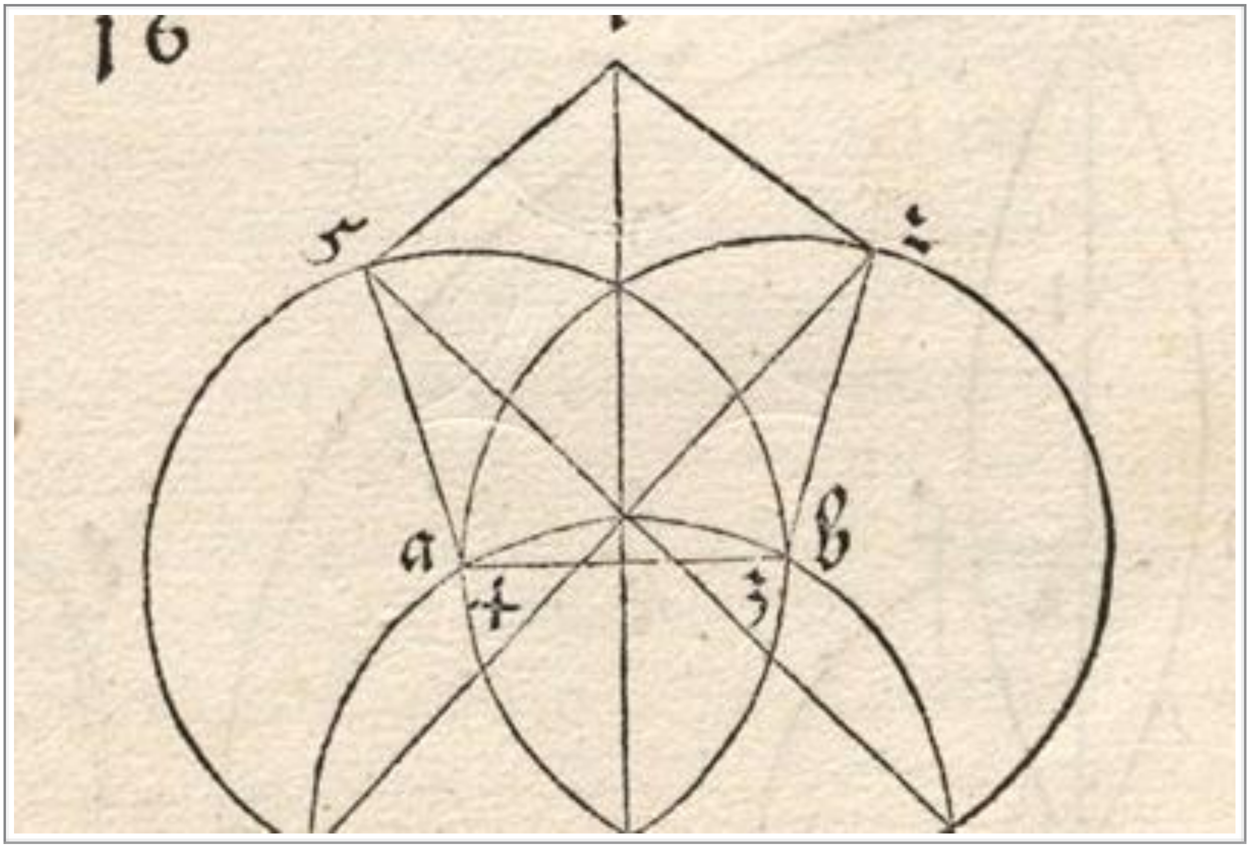
Great Hearts

The Teaching of Mathematics



New Faculty Orientation

Summer 2016

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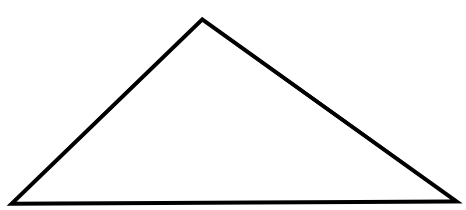
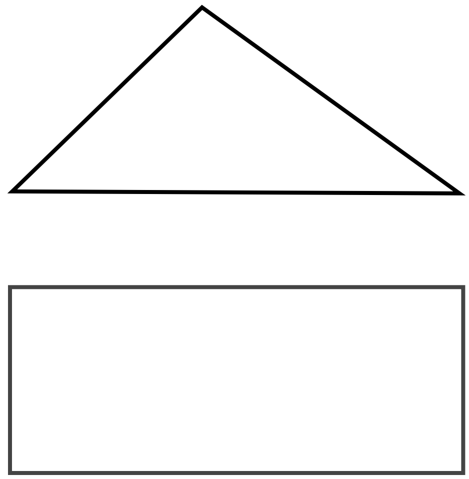
The Approach to the Discipline

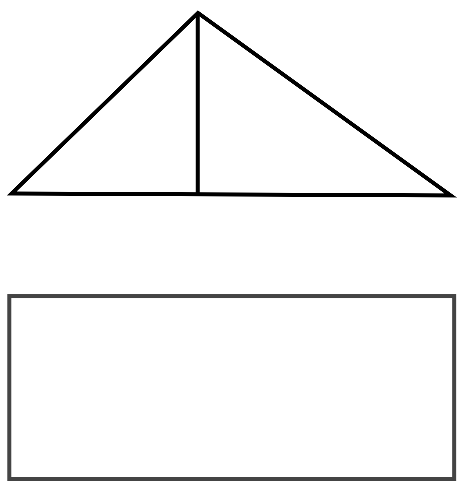
We begin by considering the very form of the discipline, specifically how the Trivium (grammar, logic, rhetoric) are employed in the mathematical endeavor. Like all disciplines, mathematical exploration is the pursuit of truth. “What kind of truth” we will consider in a moment, but let us first examine the form of the pursuit itself. The mathematician is a professional “proof writer.” Euclid, in his presentation of geometry, is a foundational for understanding this process. A formal mathematical system builds upon first principles (“postulates” or “axioms”) and definitions. From there, the mathematician makes conjectures about mathematical truths and proceeds to offer step-by-step logical proofs for these conjectures. The final proofs are the “product” of the *art* of mathematics. The investigation, conjecture, and writing of proofs is the very *doing* of mathematics. Therefore, proofs should be present at *all* level of mathematics education, though the level of detail and formality will vary depending on the student’s mathematical development.

Since most of us have experience with proof at the high school stage, likely through the teaching of Geometry, we will instead illustrate the concept of proof development with a straightforward example that can be used in the elementary years. For this example, we will presume that students have already examined the formula for the area of a rectangle (being the product of the length and the height). We are asked to think deeply about the area of a triangle.





In order to help solve the problem, we think "inside the box" on this one.
The question is, what fraction of the area of the box is taken up by the triangle? If it



is not immediately clear, a simple auxiliary line can help us.
We should be able to see that the triangle is exactly one half of the rectangle. Because we know that the area of the rectangle is the product of the length and the height, and because the triangle shares the same length and height, it should be clear that the area of the triangle is one half the product of the length and the height.

Notice that every step in this process proceeds logically, and notice that the proof not only demonstrates *that* the fact is true, but also *why* the fact is true. This is the mark of a well-written proof. Notice also that the use of logic, the second leg of the

Trivium, is central to this process. In fact, it is in the study of mathematics that students encounter most directly the study of logic. The essence of a proof is a movement of logic.

Yet the proof proceeds also as a narrative. It is incumbent upon the mathematician not only to *know* the truth and to have *proven* the truth for himself, but also to communicate that truth to an audience. Therefore, the use of *grammar* and *rhetoric* are also part of the very form of mathematics. The very act of proof writing is not only an exercise in logic, but it equally an exercise in *rhetoric.*

Notice that the proof of the above theorem about the area of a triangle is not in the typical "two-column" format that many of us were taught in high school geometry. The two-column format can be a good organizational, and therefore pedagogical, technique, yet a mature mathematical proof is always written as prose. The prose proof contains no less details than the two-column format, but it *reads better*.

Now, the above proof is sufficient for an elementary student. I used this as an illustration for that very reason. For a professional mathematician, or even for a more mature mathematical student, there are many details to fill in. *Why* is the triangle half of the rectangle? Can I add areas together? Several other questions will arise are we press into the issue in more details. As we ask more "why" questions, we will produce a more mature, or what mathematicians call more "rigorous", proof. In Euclid, as in other examples of mathematics, we encounter the "rigorous proof", one in which all of the *why* questions have been answered.

Seeing proof writing from this "top down" perspective, wherein we first grasp the basic narrative of the proof (in this case we grasp the fact that a triangle has area of one half the product of the base and the height because it can be boxed in by a rectangle that is equal in base and height) and then proceed to fill in the details by asking, "How do we know that?" for each step, is a very different method than most of us were taught in high school geometry. The typical method is, "What do I know, and where can I go from here?" The problem with this later pedagogy is that a student may end up with a correct proof, and may even understand every step, but there is no guarantee that the proof discloses the truth of being, i.e. there is no guarantee that the student understands *why* the theorem is true, only *that* it is true. It is a little like making our way successfully through a corn maze, understanding *that* we escaped, but not fully grasping *how* we escaped, other than a series of correct turns that landed us at the exit. The emphasis on narrative is like looking at the max from above. The top down process of emphasizing the narrative of the proof first also squares (pun intended) very nicely with the Socratic method. Once a student has demonstrated why a truth holds, we then press into their reasoning by asking for it to be clarified. What comes out is a rigorous mathematical proof complete with the entire "step-reason" framework for which the two-column version is so famous.

Before moving on from the Trivium, we should note that mathematics, while making use of "grammar" understood in the specific sense of using words properly to communicate truth, also has something of its own grammar, e.g. variables, operations, equations, etc. This is not in contradiction or even something separate to the grammar of words, but rather sits along side of it so as to be more efficient in communicating. It is easier to write "$2x + 1$" rather than "the product of two and a number increased by the value of one." As a side note, we should avoid referring to mathematics as "a language." Mathematics *has* a language (vocabulary, symbols, etc.), much as do the other disciplines. Yet it is much more than a language. It is a *discipline*, an area of study, in which students strive to understand the truth of pure forms.

If the process of conjecture and proof is the form of the discipline, what is the "matter" of mathematics? The ancients and medievals understood this well in their categorization of the quadrivium: mathematics is the study of number (the discrete) and shape (the continuous). (They also considered music, as quantity in motion, and astronomy, as geometry in motion, as part of mathematics.) The "stuff" that we deal with is the very nature of numbers and the very nature of shapes.

Beginning at the End

The “end” of mathematics is, of course, the formation of the human soul. Like every other discipline, there is something unique in the human soul that can only be satisfied by thinking mathematically and by knowing mathematical truths. But for a K-12 curriculum, we must decide that “proximate end”, that is the point to which we want all students to arrive by the time they leave us. It goes without saying that we want them to understand the very form of mathematics. That is, we want them to be able to think mathematically. The first section, on proof, was dedicated to this. Here, however, we are asking what the “content” end is for our students. For the K-12 curriculum that answer is Calculus. We say this for a few reasons. First, Calculus will be the student’s first experience with higher level mathematics. While Euclid is challenging and beautiful and essential for students to learn, it is really only the foundation (or one aspect of the foundation) for high level mathematical thought. Calculus is something different. It is on a higher plane. Calculus is one of the great pinnacles of Western mathematics. It has a particular value for K-12 education in that it beautifully brings together the two realities that students have been studying: number and shape. If “algebra” is understood as the proper abstraction of *number*, Calculus brings together algebra and geometry. For that reason, it is a beautiful capstone for the study of mathematics at Great Hearts.

The K-5 Mathematics Curriculum

With that, we are ready to talk through the progression of mathematics from Kindergarten through Calculus.

At the K-5 academies, our students study Singapore mathematics. This is not the place to go into the details and history of Singapore. It is sufficient for us to highlight the main goals that we have for our K-5 students in their mathematical training, and it is enough to know that the Singapore curriculum helps us in accomplishing these goals.

Students’ primary goal in grades K-5 is to become familiar with basic numbers and shapes and the way that they act together. As philosophical realists, we believe that truth can be encountered and known, and that students at early ages learn best by examining what is close to them. For this reason, mathematics in the K-5 academies is taught with heavy reliance on manipulatives: physical representations of numbers. When students learn to count, they will count actual objects. Through a deliberate program of using these manipulatives, students begin their journey towards abstraction. They move from the physical, concrete objects, towards a pictorial representation of the objects, and finally on towards the abstract symbols and algorithms for operating on these symbols. We do this not only to aide their understanding, but also to emphasize that *mathematics is grounded in the real*.

The curriculum at the K-5 level focuses on developing number sense. For this reason, students are not given calculators to use. Instead, they are taught various techniques for taking apart numbers and putting them back together so as to be able to compute complicated calculations mentally. We do this *not* for utilitarian purposes, as if to say, "Someday you may have to do a calculation without a calculator in your hands, and what then?" but rather we do this because the process of disassembling and reassembling numbers is the *only way* to teach students the very *nature of numbers themselves*. That is precisely what we mean by "number sense."

In order to help students with their number sense, we also emphasize the need for memorized math facts. There are certain things that everyone should have at the tips of their fingers. If basic facts are not immediately accessible, learning more complicated truths proves harder than it needs to be. There has been a pendulum swing in the last fifty years with regards to mathematics and memorization. Several decades ago, students were taught simply to memorize without understanding. Therefore, many people grew up being able to *perform* standard algorithms, but having no mathematical understanding of why the algorithm worked. This is not noble. But in recent decades, the pendulum has swung the opposite direction, now claiming that memorization is not at all needed and that knowing *why* is the only important thing. This is also not noble. This is to miss that point that the memorized facts themselves *aide* understanding, they do not hinder it.

The same pendulum has swung with regards to the standard algorithms. On the one hand, there are those that would teach only these algorithms (column addition, long division, etc.) absent of any explanation, and on the other hand there are those that falsely claim, "All methods of solving a problem are equally valid and should be honored." In our K-5 mathematics program, we want student to be able to think through the nature of the numbers in order to arrive at answers in creative and varied ways, but we *also* understand that the standard algorithms are beautiful in their elegant efficiency, and they deserve a pride of place among techniques for operating on numbers. Yet we teach these algorithms with the goal that students will *understand* them. In order to help with this task, we emphasize to an almost obsessive degree the concept of *place value*. Here we see the most obvious example in the K-5 curriculum of mathematics having a *particular grammar*. The very way that we write numbers in the base ten system is an organized system of combining symbols for the sake of understanding and communicating the reality of numbers themselves.

The Middle School Mathematics Curriculum

In the middle school, if we have done out jobs right at the K-5 level, we have present students who are ready to think about numbers on a more abstract level. This is why a Pre-Algebra class is the perfect time to study negative numbers. These are not physical realities in the same way that positive integers are. (This is not to say that negative number cannot be used to model physical phenomena, e.g. velocity, but merely to suggest that one cannot "hold" negative three apples in one's hand.)

Yet in a beautiful way, the old abstract (the "number itself") form the K-5 courses, becomes the new concrete. The new abstract is an abstraction of the number itself, i.e. the "variable". By the end of Pre-Algebra, students are ready to understand the concept of a variable. Yet we always bring them back to the concrete. When discussing algebraic properties, e.g. "$a(b + c) = ab + ac$", we ask students to substitute values over and over again to "check to see if it is true." Students will be tempted to disassociate algebraic properties from numbers themselves. We see this most often when the "invent" their won property or misapply a familiar property. When deriving properties, we ask students to act like mathematicians: play with the numbers, combine them in various ways to see if a property emerges. *Then* ask if this property always holds true for all numbers. The "number" now plays a similar role to the manipulative at the K-5 level. It is the *particular,* and the property is the *general* or the *abstract.* In this way, algebraic truths are never presented as mere conventions or "rules" upon which mathematicians decided, but rather these facts are *true*. Algebra, as abstract as it can seem to students, is grounded in the real.

The most important aspects of the middle school Algebra curriculum are (1) the solving of equations and (2) the graphing of equations. With regards to the later, we emphasize the genius of the Cartesian plane in uniting equations with pictures (numbers with geometry), and thereby laying important groundwork for the future study of Calculus. We repeat incessantly the truth that "points that solve the equation are on the graph, and points that are on the graph will solve the equation." The picture, i.e. the graph, is never something *different from* the equation. It *is* the equation in a different representational form. For this reason, we want students to plot equations by using tables of values almost incessantly (beginning actually towards the end of Pre-Algebra) rather than jumping too quickly to the "quick" methods (e.g., slope-intercept form of a line). Jumping too quickly to the latter risks students not remembering that the graph is nothing more than the infinite collection of points (ordered pairs) that solve the equation. For this reason, too, we do not allow graphing calculators in the student's first experience with Algebra.

With regards to solving equations, we begin by having students study and master *linear* equations. They quickly learn to solve (and graph) the entire set of these equations. If it is taught effectively, students should almost get the sense that, "This whole algebra thing is *easy*. We seem to be able to solve *any* equation." It is at this point that they are ready for quadratic equations. It is here that students realize how *difficult* these equations can be. They struggle by solving only certain categories of quadratics, beginning with those having only a quadratic term and not a linear term (so factoring is not necessary). Even here, these equations are far more interesting than the linear counterparts if only for the reason that they often have *two* solutions.

And yet, introducing a linear term into the quadratics makes these equations considerably more difficult.

At this point the technique of factoring is introduced, and a whole new category can now ingeniously be tackled. The beauty and creativity of this method should never be understated. And yet there are so many that remain “difficult” to do, i.e. those that do not factor. Finally, we introduce the concept of completing the square, and students can now solve (and graph) “any” quadratic equation. They have matured mathematically, and they have acted like mathematicians in doing so. What seemed difficult has proven, in fact, *to be difficult*, but it has not proven to be impossible.

The culmination of this curriculum is the development of the quadratic formula itself. Students should be responsible for reproducing this derivation, for it is a marvel of equation solving. The entire solution for this difficult set of problems is encapsulated in this single beautiful formula.

The High School Mathematics Curriculum

At this point, students are ready for a much more mature treatment of the discipline. Geometry is a body of knowledge concerned with mathematical space. It is organized and systematic, beginning with certain definitions, postulates, and axioms, and deducing from them other truths which are then called theorems. It is certain and unchanging, more certain than the daily rising and setting of the sun. For this reason it is a science.

Geometry is also an art, inasmuch as it involves the creation of mathematical forms (lines and shapes), but even more importantly, because it involves the creation of proofs. Any given theorem in geometry can be proven in many ways, but among these there are some proofs which are more elegant than others. A well done geometric proof is among the most beautiful of human creations.

This particular geometry course is a study of Euclid’s Elements, the first and greatest textbook of geometry. This book is among the most beautiful and intelligible works of geometry ever written. Euclid is single handedly responsible for compiling and ordering the geometric discoveries of the ancient world into a single, organized geometry text. From the 2nd century B.C. through the early 20th century, the Elements has stood firmly as the textbook for all students of geometry. Abraham Lincoln, for instance, kept of a copy of the Elements in his saddlebag and studied proofs in his evenings before bed. The Elements has proven itself to be the ideal geometry text for the classical, liberal arts mathematics class.

As a proof-centered geometry this course is a student's first experience with a mature and rigorous mathematics course. This is precisely why we place this course at the start of the high school mathematical experience. The first goal is to master the core definitions, postulates, and propositions of Euclid. Mastery of these involves the ability to recollect what they are and the ability to rigorously prove the theorems.

In the tenth grade, students will pick up their study of algebra once again, but this time with a much more mature view of mathematics. The tenth grade course is a course in the study of functions. It presents again a formal mathematical system, much like the study of Euclid, but it does so in the realm of algebra. Because students are now equipped with the tools of proof, they are able to grapple with more sophisticated realities in algebra, both as equations and as graphs. This return to algebra, this deep dive into the world of functions, sets the students up for their two-year study of Calculus.

All Great Hearts students are treated to the formal study of Calculus. We take students through the two famous problems: the slope of a tangent line and the area under the curve. When it is done correctly the revelation of the Fundamental Theorem of Calculus is the centerpiece of the Calculus experience. To learn that these two fundamental problems are intimately related is a beautiful experience. The *culmination* of Calculus, however, occurs in the study of Taylor's Theorem. Here, students bring everything they know about the most important class of functions from their tenth grade study, i.e. polynomials, and situate them as the real building blocks of *all functions*.

Through all of this, students are treated to Cauchy's original work on Differential and Integral Calculus. Much like Euclid provided insight into geometry as a human endeavor, Cauchy does the same for Calculus.

A Sense of Wonder: The Study of the Forms

After taking this tour through the content, we should not lose sight of the fact that we aim to teach students to think, and to *wonder* mathematically. Every teacher of mathematics should be asking themselves interesting mathematical questions, even if such question are beyond the scope of what they may ask of their students. For examples, an elementary teacher, when teaching problems such as "73 – 28" should ask, "Why do we regroup 10? The problem here is that 8 is greater than 3, but it is not 10 greater than 3. It is only 5 greater than 3. So I only really need to regroup 5? Will that work?" Or perhaps the same teacher might ask, "Actually, I *can* subtract 8 from 3, it's just that I get '–5' in the one's place. Is that correct?" These are questions that might, or likely would, confuse a second grader, and yet they are beautiful mathematical questions about which the second grade teacher should wonder.

A similar question from geometry might be, "I have noticed that regular hexagons fit nicely together as a tile pattern. [Mathematicians say that these shapes 'tile the plane'.] What other regular polygons will do this?" A Calculus teacher might ask, "I know that a function must be continuous to be differentiable, but it need not be differentiable in order to be continuous. I wonder if there is a function that is continuous *everywhere* but differentiable *nowhere*."

These are questions of wonder, and they demonstrate that *wonder* is in *act*, not a passive experience. We choose to wonder, and when we do so, we communicate that same wonder and same love of the discipline to our students.

From Kindergarten through Calculus, we should not lose sight of the fact that students, in their study of mathematics, are studying the *pure forms of the heavens*. The poet, Edna St. Vincent Millay, understood this in specific reference to Euclid. Whether or not "Euclid *alone* has looked on Beauty bare" may be up for discussion, but it seems at least clear that those who strive to internalize mathematic truths, be they of arithmetic, geometry, algebra, or calculus, will certainly "have heard her massive sandal set on stone."

Euclid alone has looked on Beauty bare
By Edna St. Vincent Millay (1923)

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.